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Úloha předpokladu linearity v Nashových hrách

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BACHELOR THESIS

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On the Role of Linearity Assumption in Nash Games

Department of Probability and Mathematical Statistics

Supervisor: RNDr. Michal Červinka Ph.D.

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In this place I would like to thank my supervisor for his time, patience and tireless help during writing this thesis.

Hereby I declare that I compiled this bachelor thesis independently, using only the listed literature and resources. I agree with lending and publication of this thesis.

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Abstract: In the bachelor thesis On the Role of Linearity Assumption in Nash Games we discuss the concept of the game theory and Nash equilibria. We briefly introduce finite and continuous games. Further, we study assumptions about payoff functions of players in continuous games and make conclusions about their impact on an existence and calculation of Nash equilibria. In the main focus there are affine linear, piecewise affine linear and (strictly) concave payoff functions. Each case is illustrated in an appropriate example.

Key words: game theory, Nash equilibrium, payoff function

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Abstrakt: V bakalářské práci Úloha předpokladu linearity v Nashových hrách se zabýváme konceptem teorie her a Nashovými ekvilibrii. Stručně představujeme spojité hry. Dále studujeme předpoklady kladené na výplatní funkce hráčů ve spojitých hrách a zhodnocujeme jejich vliv na existenci a výpočet Nashových ekvilibrií. Důraz je kladen na lineární, po částech lineární a (ryze) konkávní výplatní funkce. Každý případ je ilustrován na vhodném příkladě spojité hry.

Klíčová slova: teorie her, Nashovo ekvilibrium, výplatní funkce

Contents

Chapter 1

Introduction

When the term of *a game* is said, many readers will probably imagine a poker game, the chess or some computer game for example. All of these popular games have something in common. It is the situation where players are making decisions and simultaneously are influencing decisions of other players in order to win. This general decision-making situation is the basis of the term *a game* in the game theory. In this elementary sense, we are playing games more often than we in fact realize. We are playing a game every time we get in an interaction with other people and try to make for ourselves the best decisions.

In context of this interpretation, *a player* in the game theory is anyone or anything that is able to make decisions. Hence it does not have to be only human, but also an animal, a company or an institution for example. In every game the decision-maker has at disposal some *strategies* he can choose. Now, the reader is probably imagining some ways or methods which lead to the win. Such reader is not far from the truth. In the game theory, player's *strategies* are all decisions that are available to him in the moment of his decision-taking. It is common in games that each player can clasify strategies. He is able to determine which strategy is more or less helpful for him. Hence there is defined some *evaluation function*. This function has several names, for example *cost function*, *utility function*, *payoff function* and others and it is matter of taste which one is used.

One of the important ingredients of the game theory is a concept of game equilibria. It is a problem of searching stable points, which mean situations players can occur in and which are in some sense stable. One type of such points is *a Nash equilibrium*. It is named after mathematician John Forbes Nash who won the John Von Neumann Theory Prize in 1978 for this contribution to the game theory. This point is stable in sense that no player wants to change his strategies on his own because any deviation would make him worse off.

In examinations of such equilibrium points there are often made some simplifying assumptions about evaluation functions which help to make these problems easier to solve. The purpose of this work is to examine impacts of such assumptions on an existence and calculating Nash equilibria. We chose several of such common assumptions. They are linearity, piecewise linearity and (strict) concavity.

Chapter 2

Game Theory

Suppose we have a set of N players. For every player i let us define a set of strategies $S_i \in \mathbb{R}$, $i = 1, \ldots, N$. Then $S = S_1 \times ... \times S_N$ denotes a set of strategies of all players. The strategy selected (played) by player i will be denoted $s_i \in S_i$. Let $f_i : S \to \mathbb{R}$ be a real-valued function of N variables. Function f_i will be called *a payoff function* of player i and its value will represent a payoff of player i if all players choose a vector of strategies $(s_1, ..., s_N) \in S$.

Definition 1. The *game in normal (strategic) form* is defined as the triplet $(\{1, ..., N\}, S, f)$, where the set $\{1, ..., N\}$ represents a set of N players, S is a set of strategies of all players and $f = (f_1, ..., f_N)$ is an N-dimensional real-valued function $f : S \to \mathbb{R}^N$ where f_i is a payoff function of player i for every $i \in \{1, ..., N\}$.

A payoff function associates strategies with utilities gained by a player in a game and hence it sets preference relations on the set of strategies. These preference relations differing from player to player are described by the following definition.

Definition 2. We will say that vector of strategies $(s_1, ..., s_N)$ is *at least as good as* $(s_1^{'}, ..., s_N^{'})$ to Player i if and only if

$$f_i(s_1, ..., s_N) \geq f_i(s_1^{'}, ..., s_N^{'}).$$

Further, we will say that Player i *prefers* a vector of strategies $(s_1, ..., s_N)$ to $(s_1^{'}, ..., s_N^{'})$ if and only if

$$f_i(s_1, ..., s_N) > f_i(s_1^{'}, ..., s_N^{'}).$$

To be able to make any conclusions we need to know players' behavior in games or at least to expect some. Hence there is a general assumption about players' behavior made in the game theory which is believed to describe reality closely. It is the assumption of *rationality*.

Definition 3. We say that Player i *behaves rationally* (*is being rational*) if he chooses a vector of strategies $s = (s_1, ..., s_N) \in S$ while there is no other available vector of strategies $t = (t_1, ..., t_N) \in S$ which could be preffered to s.

We will suppose in the following that all players behave rationally. Clearly from definition, player i behaves rationally if and only if he chooses such vector of strategies that maximizes his payoff function.

Now let us give a general classification of games used in the literature which is mainly according to the properties of the triplet $(\{1,...,N\}, S, f)$:

- Size of the set of players: When having N players we generally talk about N*-person* games.
- Cardinality of the set S: When number of element of S is finite, we talk about *finite* games. Otherwise games are called *infinite* or *continuous.*
- Relation between players' payoffs: When

  $$\sum_{i=1}^{N} f_i(s_1, \ldots, s_N) = 0$$

  holds, we talk about *zero-sum* games, otherwise they are called *nonzero-sum.*
- Cooperation between players: If players are allowed to cooperate, i.e. to change information during a game, we say that they play a *cooperative* game. In the other case the game is called *non-cooperative.*
- Repetition of a game: If a game is played only once which means that players chooses their strategies only once, we call this game *static.* If a game is repeated we call it a *dynamic* game.

In the center of focus of this work there are N-person static non-cooperative nonzero-sum games, both finite and continuous. The other types of games are not in our interest and hence we omit their proper description.

In the following sections we give a brief introduction to finite and continuous games.

2.1 Finite Games

Let us have a finite set of strategies

$$S_i = \{s_1, \ldots, s_{m_i}\}, \quad m_i \in \mathbb{R},$$

for every $i \in \{1, ..., N\}$.

Suppose that every strategy from the set $\{s_1, \ldots, s_{m_i}\}$ is chosen by player i with some probability. Then to every set S_i we can assign a vector

$$p^i = (p^i_1, \ldots, p^i_{m_i}),$$

where p^i_j is a probability of choosing strategy s_j by player i and

$$\sum_{k=1}^{m_i} p^i_k = 1.$$

Definition 4. A vector p^i is called a *mixed strategy* of player i in finite games. If there exists $k \in \{1, ..., m_i\}$ such that $p^i_k = 1$, then we say that player i plays a *pure strategy.*

If player plays pure strategy, he chooses a single strategy, whereas if he plays a mixed strategy, he chooses the probabilities with which he plays all his strategies. Now, the reader might be convinced that playing mixed strategy could be more advantageous than playing pure strategy. If a player plays a pure strategy, opponents could discover it and profit from this information or, moreover, play strategies that would be harmful for this player. On the other hand if player plays mixed strategy, he protects himself from discovering his played strategy since he does not know it himself. However, let us understand that the game would have to repeat so that opponents could discover player's pure strategy. Moreover, there are games where player does not care anyway if opponents know his strategy. It is for example the case where players maximize their payoffs independently on strategies of their opponents.

Since this work is interested in static games where strategies of players are not known to their opponents, we focus on pure strategies.

Let us return to finite games in general. There are two most common representations which help to model finite games. In the rest of this section we give an overview of them.

The *matrix form* is a representation used for 2-person finite games. It is based on an $(m_1 \times m_2)$-dimensional matrix

$$A = \{(a_{ij}, b_{ij})\}_{\substack{i=1,\dots,m_1\\ j=1,\dots,m_2}},$$

with elements (a_{ij}, b_{ij}), where a_{ij} and b_{ij} are payoffs of player 1 and player 2 if player 1 chooses strategy s_i and player 2 chooses strategy s_j.

In the Example 1 we show a game in the matrix form and, moreover, we describe player's rational behavior.

Example 1:

Suppose we have two players. For the purpose of this example and other examples in this chapter player 1 will be called Kane and player 2 will be called Loli. They are playing a game with a finite set of strategies represented by the following matrix form.

Loli

	(1,6)	(2,2)	(3,5)
Kane	(4,5)	(5,4)	(6,2)
	(3,4)	(7,5)	(3,6)

The strategies of Kane are rows $\{1, 2, 3\}$, the strategies of Loli are columns $\{1, 2, 3\}$.

Let us illustrate, what Loli's rational behavior means. If Kane chooses row 1, Loli will choose a column that would lead to her maximum payoff. In this case it is column 1 and her payoff will be 6. If Kane chooses row 2, Loli will choose again column 1 and her payoff will be 5. If Kane chooses row 3, Loli will choose column 3 and her payoff will be again 6. Loli will never choose column 2 because she would not behave rational.

△

The *extensive form* is a representation used for N-person games in general. It is based on a tree structure where roots are positions of players before their move and branches represent strategies available in that position (root). This representation is suitable for dynamic games because of its ability to illustrate sequences of players' moves in time.

A game in an extensive form is illustrated in the following example.

Example 2: (Battle of Sexes)

Kane and Loli want to spend an evening together. Kane would like to go to the car-racing whereas Loli would like to go to the theatre. They did not see each other for a whole day and have no possibility to let each other know where they decided to go. However, Kane knows that Loli wanted to go to the theatre and Loli knows that Kane wanted to go to the car-racing. Now, both of them have to decide where to go.

Let us suppose that their payoffs are evaluated as follows:

- If they decide to go on different places, both will be disappointed and their payoffs will be 0.
- If both decide to go to the car-racing, Kane will be happy and his payoff will be 10. Loli will be glad that she will have met Kane but not as happy as she would be at the theatre with him and her payoff will be 5.
- If both decide to go to the theatre, Loli will be happy and her payoff will be 10 whereas Kane's payoff will be just 5.

We can illustrate this game in the following extensive form.

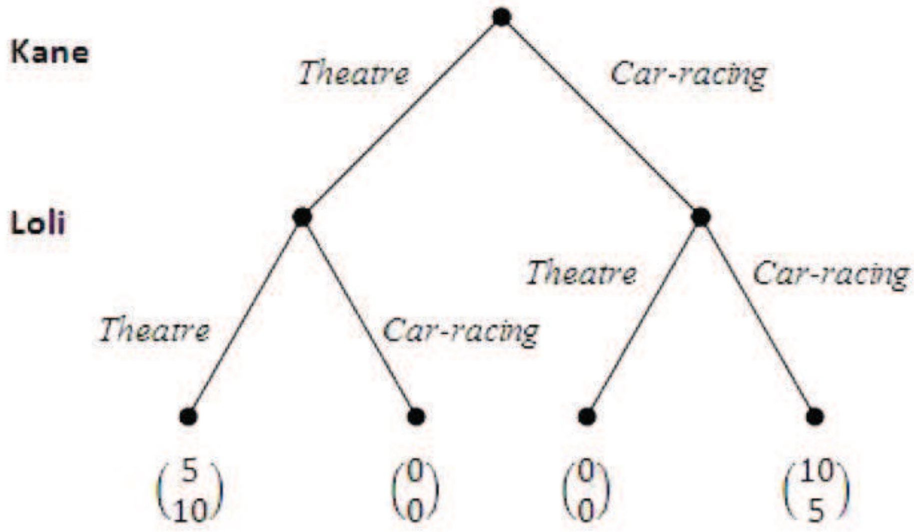

Figure 2.1: Battle of Sexes

Loli does not know where Kane has gone. Hence she does not know her exact position (root) she is occurring in. (In the Figure 1.1 these two roots are illustrated tied together.) If Loli would know where Kane has gone, she would know her position and also which subtree she should choose. In this situation she would always make the same choice as Kane because it is rational behavior for her. This game is generally known as *Battle of Sexes*.

△

2.2 Continuous Games

In continuous games player i chooses from an infinite set of strategies S_i. In the beginning of this chapter we defined S_i as a subset of $\mathbb{R}$. S_i are often considered to be convex in the game theory and we will make this assumption as well. Hence we will represent them by convex intervals from $\mathbb{R}$.

In general, there need not be any restrictions on payoff functions in continuous games. However, in the next chapter we make an assumption of continuity.

Although we said that pure strategies are in our interest, for completeness we define also mixed strategies in continuous games. Hence, we need to define some representation which would describe a probability to every point from S_i.

Definition 5. The probability measure μ_i on S_i is called a *mixed strategy* of Player i in continuous games.

In the next chapter we will omit finite games and focus on continuous games. It is because impacts of assumptions about payoff function are more interesting to examine on infinite convex sets of strategies.

In the center of this work there are impacts on an existence and finding so-called *Nash equilibria*. The concept of *Nash equilibria* is introduced in the following section as well as one other type of equilibrium points, so called *Pareto optima*.

2.3 Equilibria

The concept of equilibrium points given in this section is generally same for finite and continuous games.

For simplicity let us denote by $s_{-i} \in \mathbb{R}^{N-1}$ a vector $(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$ for every $i \in \{1, \dots, N\}$. It means that s_{-i} is a vector of strategies of all players except of player i.

Further denote by

$$S_{-i} := \bigtimes_{\substack{j=1 \\ j \neq i}}^{N} S_j$$

a set of all strategies of all players except of player i.

Definition 6. Define a *value function* $f_i^M : S_{-i} \to \mathbb{R}$ subsequently:

$$f_i^M(s_{-i}) := \sup_{s_i \in S_i} f_i(s_i, s_{-i})$$

for every $i \in \{1, \dots, N\}$.

Value of function f_i^M determines the maximum payoff which Player i can get if other players play a vector of strategies s_{-i}.

Definition 7. Define a set $C_i(s_{-i})$ subsequently:

$$C_i(s_{-i}) := \{s_i \in S_i \mid f_i(s_i, s_{-i}) = f_i^M(s_{-i})\}$$

for every $i \in \{1, \dots, N\}$. Then $C_i(s_{-i})$ is called *the optimal response set* or *the rational reaction set* of player i. If $C_i(s_{-i})$ is singleton for every vector s_{-i}, then it is called *the reaction curve* of player i.

According to Definition 7, $C_i(s_{-i})$ can be understood as a set of all strategies of player i which lead to the maximum payoff of player i while other players choose a vector of strategies s_{-i}.

Definition 8. A vector of strategies $\bar{s} = (\bar{s}_1, \dots, \bar{s}_N)$ is called *a Nash equilibrium* if it holds

$$\bar{s}_i \in C_i(\bar{s}_{-i})$$

for every $i \in \{1, \dots, N\}$.

Let us note that it holds that $\bar{s}_i \in C_i(\bar{s}_{-i})$ if and only if

$$f_i(\bar{s}_i, \bar{s}_{-i}) = f_i^M(\bar{s}_{-i}) = \sup_{s_i \in S_i} f_i(s_i, \bar{s}_{-i}).$$

Hence a vector of strategies $\bar{s}$ is a Nash equilibrium if a strategy played by every player leads to his maximum possible payoff when the strategies of other players are fixed. Nash equilibrium is a stable point of a game in sense that no player wants to deviate from his strategy on his own because he would not better himself. It is also called *a non-cooperative equilibrium* of a game. However, let us mention that it doesn't have to be necessarily a point such that players could not better themselves by changing strategies of more than one player, for example by cooperating in choosing their strategies. In other words, it does not have to be so-called *Pareto optimal. Pareto optimum* is another equilibrium point of games. Let us see the next definition.

Definition 9. A vector of strategies $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_N)$ is called *Pareto optimal* if there exists no vector $t = (t_1, \ldots, t_N)$, $t \neq s$, such that

$$f_i(t) \geq f_i(\tilde{s}), \text{ for every } i \in \{1, \ldots, N\},$$

and simultaneously

$$f_j(t) > f_j(\tilde{s}), \text{ for some } j \in \{1, \ldots, N\}.$$

Pareto optimum is a point where no player can better himself without another player getting worse off it. This point can be found as an equilibrium point in cooperative games if players maximize a sum of their payoffs. However, there can be many Pareto optima in cooperative games but not every one has to be an equilibrium point in sense that not every one has to maximize a sum of players' payoffs. To learn more about cooperative games or Pareto optima, reader can see [1] and [2].

Now let us return to the concept of Nash equilibria. Let us denote by

$$C : (s_1, ..., s_N) \mapsto (C_1(s_{-1}), ..., C_N(s_{-N}))$$

a mapping which assigns to every vector of strategies the optimal response sets of all players. Clearly, a Nash equilibrium is a fixed point of mapping C. Let us remind that $\bar{s}$ is a fixed point of mapping C if it holds that $\bar{s} \in C(\bar{s})$.

Hence the existence of a Nash equilibrium is equivalent to the existence of a fixed point. This leads us to the necessity of an application of fixed-point theorems. We will introduced two of the most used ones.

Theorem 1 (Brouwer's Fixed-Point)**.** *If K is a compact and convex subset of a finite-dimensional space and $C : K \to K$ is a continuous mapping, then C has a fixed point, i.e. there exists $k \in K$ such that $C(k) = k$.*

Theorem 2 (Kakutani's Fixed-Point)**.** *If K is a compact and convex subset of a finite-dimensional space and $C : K \to K$ is an upper semicontinuous mapping, which assigns to each $k \in K$ a nonempty, closed, convex subset of K, then C has a fixed point.*

There is a very important conclusion in the game theory based on fixed-point theorem. The author of this theorem is J. F. Nash.

Theorem 3. *Every N-person finite game has a Nash equilibrium in mixed strategies.*

Proof. The statement follows from Theorem 1.4.1. □

Fixed point theorems are useful tools for proving an existence of Nash equilibria in games. However, they do not provide any instruction how to calculate them.

From Definition 8 it follows clearly that Nash equilibria can be found as intersection points of the optimal response sets of all players.

Let us note that

$$C_i(s_{-i}) = \{t_i \in S_i \mid f_i(t_i, s_{-i}) = \sup_{s_i \in S_i} f_i(s_i, s_{-i})\} = \{\arg \sup_{s_i \in S_i} f_i(s_i, s_{-i})\}.$$

One convenient way of finding such sets is to let the players optimize their own criterions assuming that strategies of their opponent are fixed. Hence we need to introduce the functions

$$f_i^M(\bar{s}_{-i}) = \sup_{s_i \in S_i} f_i(s_i, \bar{s}_{-i})$$

for every player i and further, Nash equilibria can be found as solutions of following equations

$$f_i(\bar{s}_i, \bar{s}_{-i}) = f_i^M(\bar{s}_{-i}), \quad i = 1, ..., N.$$

As a conclusion of this section, we summarize three cases which we will distinguish in the next chapter.

1. Nash equilibrium does not exist.
2. There exists exactly one Nash equilibrium.
3. There exist more than one Nash equilibria.

The following example illustrates computation of both Nash equilibria and Pareto optima in 2-person finite game.

Example 3: (Prisoner's dilemma)

Kane and Loli have been arrested by the police. However, there is not enough evidence against them so they are interrogated separately. Kane and Loli have two choices, to keep quiet (Q) or to report against their partner (R). Police will judge them according to the following scheme:

- If both will keep quiet, police will lock them up both in prison for 1 month.
- If one will keep quiet while the second one will report, the informer will be free and the quiet one will be locked up in prison for 10 month.
- If both of them will report agains each other, both will be locked up in prison for 5 month.

This game can be illustrated by the following matrix form.

Loli

Kane		Q	R
	Q	(-1,-1)	(-10,0)
	R	(0,-10)	(-5,-5)

Let us denote by C_1 the optimal response set of Kane and by C_2 the optimal response set of Loli.

Clearly

$$C_1(S) = \{R\},$$
$$C_1(R) = \{R\},$$
$$C_2(S) = \{R\},$$
$$C_2(R) = \{R\}.$$

We can see that for both players, Kane and Loli, it is more profitable to report against their partner no matter which strategy their partner chooses.

Since Nash equilibria can be found as intersection points of these sets, clearly, there is only one Nash equilibrium in this game. It is a vector (R, R).

This point, however, is not Pareto optimal since both players can better themselves by deviating from their strategies. According to Definition 9, points (R, S), (S, R) and (S, S) are the only Pareto optima in this game.

This example shows that Nash equilibrium does not have to be the vector of strategies that leads to the best outcome of the game in sense of maximizing a total payoff of all players.

This game is generally called *Prisoner's dilemma* in the literature.

$\triangle$

Chapter 3

Payoff Functions and Finding Nash Equilibria

In this chapter we assume the set of strategies $S_i \in \mathbb{R}$ to be nonempty and convex for every $i = 1, ..., N$. Hence we represent it by an interval from $\mathbb{R}$. For simplicity we will distinguish two different cases, closed and bounded positive intervals of real numbers $[a_i, b_i]$, $a_i, b_i > 0$, and the intervals of all real numbers $(-\infty, \infty)$. Other types of subsets are not considered in this work since we are not interested in them.

Additional assumption we make in this chapter concerns payoff functions. We will suppose them to be continuous and differentiable.

Although payoff functions of players are of N variables, players are not able to determine values of all variables on their own. They make decisions about the only one - their own strategy. All of the other variables are of other players' choice. Hence when calculating Nash equilibria we solve optimization problem of every player separately considering all strategies except of player's own strategy to be given parameters.

3.1 Affine Linear Games

Suppose we have a game of N players, where every Player i, $i = 1, ..., N$, has an affine linear continuous payoff function $f_i(s_i, s_{-i})$ on a nonempty convex set of strategies S.

We will call such game *affine linear game* (*AL game*).

In the following we will search Nash equilibria on two different types of strategies as discussed above. Firstly, we will consider the set of strategies to be compact.

A) Suppose that $S = [a_1, b_1] \times ... \times [a_N, b_N] \subset \mathbb{R}^N_+$.

From Theorem A.3 (see Appendix A) it follows that there exist coefficients $\alpha^i_1, ..., \alpha^i_N, \beta^i \in \mathbb{R}$ such that affine linear payoff function $f_i(s_i, s_{-i})$ can be expressed by the following form:

$$f_i(s_1, ..., s_N) = \alpha^i_1 s_1 + \alpha^i_2 s_2 + ... + \alpha^i_N s_N + \beta^i, \text{ for every } i = 1, ..., N, \tag{3.1}$$

where $s_i \in [a_i, b_i]$.

Since we search for the optimal response sets of player i, we need to find strategies s_i such that maximize a payoff function of player i while the elements of vector s_{-i} are given parameters. Clearly from expression (3.1), value of strategy s_i satisfying such condition does not depend on vector s_{-i} anyway. This means that player i is able to make a choice about his maximizing strategy s_i independently of choices of other players. Let us discuss this characteristic property in the following.

Let us distinguish 3 cases:

(i) $\alpha_i^i > 0$.
Hence a payoff function f_i is increasing in s_i. It follows then that the supremum is reached in only one point $s_i = b_i$. Thus for every vector s_{-i} it holds that

$$C_i(s_{-i}) = \{b_i\}.$$

(ii) $\alpha_i^i < 0$.
Hence a payoff function f_i is decreasing in s_i and the supremum is reached in only one point $s_i = a_i$. Thus for every vector s_{-i} it holds that

$$C_i(s_{-i}) = \{a_i\},$$

(iii) $\alpha_i^i = 0$.
In this case f_i does not depend on s_i at all and hence the supremum is reached in any point $s_i \in [a_i, b_i]$. Thus for every vector s_{-i} it holds that

$$C_i(s_{-i}) = [a_i, b_i],$$

Analogously we find the optimal response set of all players. We can see that their values do not depend on strategies and hence it is easy to find their intersection points.
Based on the analysis of cases (i) - (ii) we conclude that a vector of strategies $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_N)$ is a Nash equilibrium of *AL game* if for every $i = 1, ..., N$ it holds that

- $\bar{s}_i = b_i$ whenever f is increasing in its i-th component.
- $\bar{s}_i = a_i$ whenever f is decreasing in its i-th component.
- $\bar{s}_i \in [a_i, b_i]$ whenever f is constant in its i-th component.

We conclude with several obvious observations. However, let us remind that we still consider compact set of strategies.

1. There always exists a Nash equilibrium in *AL game.*
2. There exists only one Nash equilibrium in *AL game* if and only if the linear payoff function $f = (f_1, ..., f_N)$ is not constant in any component.

3. There exists infinitely many Nash equilibra in *AL game* if and only if the linear payoff function $f = (f_1, ..., f_N)$ is constant in at least one component.

Linear game and its Nash equilibria on compact set of strategies is illustrated in the following example.

Example 4:

Suppose there are only two countries on the market producing oil. Both countries have minimum required amount they want to produce. Let us denote these amounts $M_1^{min} > 0$ for country 1 and $M_2^{min} > 0$ for country 2. Both countries have also maximum possible amount they are able to produce. Let us denote these amounts $M_1^{max} > 0$ for country 1 and $M_2^{max} > 0$ for country 2, while it holds that $M_1^{max} > M_1^{min}$ and $M_2^{max} > M_2^{min}$.

The price of oil is determined by the market and countries cannot influence it. Both countries separately make decision about amount of oil they will produce. The more the country produces the more its payoff will be, however, since it decreases demand for oil produced by the other country, it decreases the payoff of the other country as well.

Let us denote by f_1 the payoff function of country 1 and by f_2 the payoff function of country 2. Further let us denote by s_1 the amount of oil produced by country 1 and by s_2 the amount of oil produced by country 2.

Hence it holds that

$$f_1(s_1, s_2) = as_1 - bs_2,$$
$$f_2(s_1, s_2) = as_2 - bs_1$$

where $s_1 \in [M_1^{min}, M_1^{max}]$, $s_2 \in [M_2^{min}, M_2^{max}]$ and $a, b > 0$.

Since payoff functions of both countries are increasing in an amount of oil produced by them, it holds that

$$C_1(s_2) = \{M_1^{max}\}, \text{ for every } s_2 \in [M_2^{min}, M_2^{max}],$$

where C_1 is the optimal response set of country 1 and

$$C_2(s_1) = \{M_2^{max}\}, \text{ for every } s_1 \in [M_1^{min}, M_1^{max}],$$

where C_2 is the optimal response set of country 2.

There exists only one intersection point of these sets hence there exists only one Nash equilibrium in this game which is (M_1^{max}, M_2^{max}).

△

Further, let us consider the set of strategies to be the whole set of real numbers.

B) Suppose that $S = \mathbb{R}^N$.

If f_i is strictly monotone on S, it does not take its supremum hence the optimal response set of player i is an empty set. If f_i is constant in s_i, it takes its supremum in every point $s_i \in S_i$ hence it holds that

$$C_i(s_{-i}) = \mathbb{R}.$$

Hence we can conclude that there exists a Nash equilibrium in *AL game* if and only if f_i is constant in s_i for every $i = 1, ..., N$. In such case, any point of S is a Nash equilibrium.

3.2 Piecewise Affine Linear Games

Suppose we have a game of N players, where every player i, $i = 1, ..., N$, has a piecewise affine linear continuous payoff function $f_i(s_i, s_{-i})$ on a nonempty convex set of strategies $S \subset \mathbb{R}^N$.

We call such game *piecewise affine linear game* (*PAL game*).

As in the case of linear games we distinguish two different types of the set of strategies in this section as well. Firstly, let us consider a compact set.

A) Suppose $S = [a_1, b_1] \times ... \times [a_N, b_N] \subset \mathbb{R}^N_+$.

Suppose there exists partition of every interval $[a_i, b_i]$, $i = 1, ..., N$, such that for every $i = 1, ..., N$ there exist points $a_i = \alpha^i_1 < \alpha^i_2 < ... < \alpha^i_{t_i} = b_i$, $t_i \in \mathbb{N}$, and that f_i is an affine linear function on

$$M_l = [\alpha^1_{l_1}, \alpha^1_{l_1+1}] \times [\alpha^2_{l_2}, \alpha^2_{l_2+1}] ... \times [\alpha^N_{l_N}, \alpha^N_{l_N+1}],$$

for every $l = (l_1, ..., l_N) \in L$, where L is a set of all N-tuples $(l_1, ..., l_N)$ such that $l_i \in \{1, ..., t_i - 1\}$ for every $i = 1, ..., N$.

Clearly, $\bigcup_{l \in L} M_l = S$ and hence f_i is piecewise affine linear function on S.

If we restrict the set of strategies S on M_l for some $l \in L$, we get a linear game and from previous section it follows that there exists at least one Nash equilibrium in such restricted game.

Let us denote by E_l such sets of Nash equilibria on the sets of strategies M_l for every $l \in L$.

The following result will help us to identify Nash equilibria in *PAL game*.

Lemma 4. *Let $\bar{s}$ be a Nash equilibrium of PAL game. Then it holds that*

$$\bar{s} \in \bigcup_{l \in L} E_l.$$

Proof. Suppose that $\bar{s}$ is a Nash equilibrium of *PAL game* and $\bar{s} \notin \bigcup_{l \in L} E_l$.

From Definition 8 it follows that

$$f_i(\bar{s}_i, \bar{s}_{-i}) = \sup_{s_i \in S_i} f_i(s_i, \bar{s}_{-i})$$

for every $i = 1, ..., N$.

Since $\bar{s} \in S$, there exists $l \in L$ such that $\bar{s} \in M_l$. Thus

$$f_i(\bar{s}_i, \bar{s}_{-i}) = \sup_{s_i} \{ f_i(s_i, \bar{s}_{-i}) \mid (s_i, \bar{s}_{-i}) \in M_l \}.$$

This means that $\bar{s}$ is a Nash equilibrium on the set of strategies restricted on M_l, which is a contradiction with $\bar{s} \notin \bigcup_{l \in L} E_l$. This completes the proof.

□

Let us make some other conclusion about *PAL game*. In the following let us fix some $i \in \{1, ..., N\}$. Then by M_l^i we will understand the set

$$M_l^i := \left[\alpha_{l_i}^i, \alpha_{l_i+1}^i\right] \times \bigtimes_{\substack{j=1 \\ j \neq i}}^{N} \left[\alpha_{l_j}^j, \alpha_{l_j+1}^j\right].$$

Further, by l_{-i} we understand the $(N-1)$-tuple $(l_1, ..., l_{i-1}, l_{i+1}, ..., l_N)$.

Suppose some interval of strategies $[\alpha_{l_i}^i, \alpha_{l_i+1}^i]$ of player i to be fixed. Since f_i is linear function in $s_i \in [\alpha_{l_i}^i, \alpha_{l_i+1}^i]$ for every vector s_{-i} and it is continuous on the whole S, it follows that f_i has a same slope in $s_i \in [\alpha_{l_i}^i, \alpha_{l_i+1}^i]$ for every vector s_{-i}. Hence player i can maximize his payoff function f_i on this interval independently of vector of strategies of other players. It means that if there exists $l_i \in \{1, ..., t_i - 1\}$ such that a function $f_i(s_i, s_{-i})$ reaches its supremum on the set M_l^i for some l_{-i}, then it reaches its supremum on M_l^i for every l_{-i}.

Hence we will fix l_{-i} and restrict the set of strategies of all players except of player i on

$$S_{-i} := \bigtimes_{\substack{j=1 \\ j \neq i}}^{N} \left[\alpha_{l_j}^j, \alpha_{l_j+1}^j\right].$$

From Theorem A.12 (Weierstrass) it follows that there exists $l_i \in \{1, ..., t_i - 1\}$ such that the payoff function takes its supremum on the set $\left[\alpha_{l_i}^i, \alpha_{l_i+1}^i\right] \times S_{-i}$. However, from assumptions about payoff functions it follows that we can find more than one of such l_i.

Denote J^i the index set of all such l_i.

For all $l = (l_1, ..., l_N)$, where $l_i \in J^i$, $i \in \{1, ..., N\}$, denote

$$H_l := \bigtimes_{i=1}^{N} \left[\alpha_{l_i}^i, \alpha_{l_i+1}^i\right].$$

Clearly, H_l is the subset of S where payoff functions of all players take their suprema. Thus if there exists a Nash equilibrium of *PAL game*, it has to be a point of H_l. Based on this and the result of Lemma 4 it follows that

$$\bigcup_l \left(H_l \bigcap E_l \mid l = (l_1, ..., l_N), l_i \in J^i\right)$$

is a set of all Nash equilibria in *PAL game*. Since there exists at least one l such that H_l is nonempty set and E_l is nonempty for all l, the set of Nash equilibria contains at least one point.

The computation of Nash equilibria in *PAL game* is illustrated in the following example.

Example 5:

Kane and Loli usually go jogging from 1 o'clock to 5 o'clock every weekend. It is Sunday and they agreed to start at 1 o'clock as usually. However, they did not agree on how long they would be running so they have to decide on their own.

Kane does not feel well today so he would not rather run longer then to 4 o'clock. He would like Loli to join him but after 2 hours of jogging he would be too tired to care whether Loli still runs with him.

Loli would like to see her favorite TV show from 1 o'clock to 3 o'clock. If she would go running, her utility from jogging will be decreasing as she will be missing her TV show. After 3 o'clock her joy from jogging will start to increase. However, she would like Kane to join her as long as he can.

Let us denote f_1 the payoff function of Kane and f_2 the payoff function of Loli. Further, let us denote s_1 and s_2 the number of hours they will be running.

It holds

$$f_1(s_1, s_2) = \begin{cases} s_1 + s_2 & \text{for } s_1 \in [0,3],\ s_2 \in [0,2] \\ s_1 + 2 & \text{for } s_1 \in [0,3],\ s_2 \in [2,4] \\ -s_1 + s_2 + 6 & \text{for } s_1 \in [3,4],\ s_2 \in [0,2] \\ -s_1 + 8 & \text{for } s_1 \in [3,4],\ s_2 \in [2,4]. \end{cases}$$

and

$$f_2(s_1, s_2) = \begin{cases} -s_2 + s_1 & \text{for } s_1 \in [0,3],\ s_2 \in [0,2] \\ s_2 + s_1 - 4 & \text{for } s_1 \in [0,3],\ s_2 \in [2,4] \\ -s_2 + s_1 & \text{for } s_1 \in [3,4],\ s_2 \in [0,2] \\ s_2 + s_1 - 4 & \text{for } s_1 \in [3,4],\ s_2 \in [2,4]. \end{cases}$$

Both functions are piecewise linear on the set of strategies $S = [0,4] \times [0,4]$ and linear on each of the subsets

$$\begin{aligned} M_1 &= [0,3] \times [0,2] \\ M_2 &= [0,3] \times [2,4] \\ M_3 &= [3,4] \times [0,2] \\ M_4 &= [3,4] \times [2,4]. \end{aligned}$$

Using the section 3.1 we find Nash equilibria on each set of strategies M_j, $j = 1,..,4$. It holds

$$\begin{aligned} E_1 &= \{(3,0)\} \\ E_2 &= \{(3,4)\} \\ E_3 &= \{(3,0)\} \\ E_4 &= \{(3,4)\}, \end{aligned}$$

where E_j is a set of Nash equilibria on the set of strategies M_j.

As the next step we fix strategies of one player and find the intervals where the payoff function of the second player takes its suprema. Kane's payoff function takes its suprema in two intervals - $[0, 3]$ and $[3, 4]$. In both intervals the supremum is reached in point 3. For Loli's payoff function there are two such intervals as well - $[0, 2]$ and $[2, 4]$. Her payoff function takes its suprema in points 0 and 4.

Clearly, there are 4 subsets of the set of strategies where Kane's and Loli's payoff functions take their suprema simultaneously. These subsets are identical to $M_1, ..., M_4$. The set of Nash equilibria E can be found as intersection points of these sets and the correspondent sets $E_1, ..., E_4$. This gives us the following conclusion.

$$E = \{(3, 0), (3, 4)\}.$$

△

In the rest of this section we discuss the *PAL game* with the set of strategies represented by the whole set of real numbers.

B) Suppose $S = \mathbb{R}^N$.

In this case piecewise affine linear continuous functions do not have to take their suprema on S and hence there do not have to exist optimal response sets. We have to put some restrictions on payoff functions. Sufficient additional assumption is to assume a function $-f$ to be coercive. One of the solvable examples is the case of concave payoff functions which are not monotone on the whole S. Such functions take their local maxima on S which are also their global maxima and hence the optimal response set of each player can be found. The problem of finding Nash equilibria for concave functions is discussed in the next section.

3.3 Concave games

Suppose a game of N players where every player i, $i = \{1, ..., N\}$, has a continuous concave payoff function $f_i(s_i, s_{-i})$ on a nonempty convex set of strategies S.

We call such game *concave game* (*C game*).

Firstly, let us consider compact set of strategies.

A) Suppose $S = [a_1, b_1] \times ... \times [a_N, b_N] \subset \mathbb{R}^N_+$.

As in the previous sections we are interested in the optimal response sets $C_i(s_{-i})$ of all players.

There is a useful theorem which describes this sets sufficiently. In the literature it is called *The Maximum Theorem* and we will present it later. First, let us give some definitions useful in the following text.

Definition 10. A mapping $P : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, which associates with each element $x \in R^n$ a nonempty subset $P(x) \subset \mathbb{R}$ will be called *a set-valued map.*

Definition 11. A set-valued map $P : x \mapsto P(x)$ is said to be
compact-valued at x if $P(x)$ is a compact set,
convex-valued at x if $P(x)$ is a convex set,
single-valued at x if a set $P(x)$ has just one point.

Definition 12. A set-valued map $P : x \mapsto \Theta(x)$ is said to be *upper-semicontinuous* (usc) at $x \in \mathbb{R}$ if for all open sets V such that $P(x) \subset V$, there exists an open set U containing x, such that for every $x' \in U \cap \mathbb{R}$ it holds $P(x') \subset V$. We say that P is usc on $\mathbb{R}$ if P is usc at each $x \in \mathbb{R}$.

Definition 13. A set-valued map $P : x \mapsto \Theta(x)$ is said to be *lower-semicontinuous* (lsc) at $x \in \mathbb{R}$ if for all open sets V such that $P(x) \cap V \neq \emptyset$, there exists an open set U containing x, such that for every $x' \in U \cap \mathbb{R}$ it holds $P(x) \cap V \neq \emptyset$. We say that P is lsc on $\mathbb{R}$ if P is lsc at each $x \in \mathbb{R}$.

Remark. A set-valued map $P : x \mapsto P(x)$ is continuous at $x \in \mathbb{R}$ if it is both usc and lsc at x.

Theorem 5 (The Maximum Theorem under Convexity)**.** *Suppose f is continuous function on $\mathbb{R} \times \mathbb{R}^n$, where $n \in \mathbb{N}$ and P is a compact-valued continuous set-valued map on $\mathbb{R}^n$. Let*

$$f^*(t) = max\{f(x,t) \mid x \in P(t)\}$$

$$P^*(t) = \arg\max\{f(x,t) \mid x \in P(t)\} = \{x \in P(t) \mid f(x,t) = f^*(t)\}$$

where $x \in \mathbb{R}$ and $t \in \mathbb{R}^n$. Then

1. *f^* is a continuous function on $\mathbb{R}^n$ and P^* is a usc set-valued map on $\mathbb{R}^n$.*
2. *If $f(x,t)$ is concave in x for each t and P is convex-valued, then P^* is a convex-valued set-valued map. When "concave" is replaced by "strictly concave", then P^* is a single-valued usc set-valued map, hence a continuous function.*

Proof. Sundaram: A First Course in Optimization Theory (p. 237). □

Denote $P : s_{-i} \in S_{-i} \mapsto [a_i, b_i]$. Then P is compact-valued, convex-valued continuous set-valued map.

From the conclusion 1 of the Maximum Theorem and using also Theorem A.12 it follows that

- $f_i^M(s_{-i}) = \sup\{f_i(s_i, s_{-i}) \mid s_i \in [a_i, b_i]\} = \max\{f_i(s_i, s_{-i}) \mid s_i \in [a_i, b_i]\}$ is a continuous function on S_{-i} and
- $C_i(s_{-i}) = \{s_i \in [a_i, b_i] \mid f_i(s_i, s_{-i}) = f_i^M(s_{-i})\}$ is a usc set-valued map on S_{-i}.

Since f_i is a concave on S, from the conclusion 2 of the Maximum Theorem it follows that

- $C_i(s_{-i})$ is a convex-valued set-valued map.

These properties of optimal response sets of all players are sufficient for proving the existence of Nash equilibria. Clearly,

$$C = (C_1(s_{-1}), ..., C_N(s_{-N}))$$

is convex-valued and compact-valued as well. It is also nonempty since f_i takes its maximum on S_i for all $i = 1, ..., N$ and hence $C_i(s_{-i})$ has at least one point.

From Theorem 2 (Kakutani's Fixed Point) then it follows that a mapping C has a fixed point. Hence there exists a Nash equilibrium in *C game.*

Further let us discuss the case of game with strictly concave payoff functions. According to the conclusion 2 of the Maximum Theorem, $C_i(s_{-i})$ is a continuous function. Hence in this case there can be used more common fixed point theorem, Theorem 1 (Brower's Fixed Point).

Finally, we can conclude that there exists at least one Nash equilibrium in C *game* for both concave and strictly concave payoff functions.

In the next part let us consider the set of strategies to be the whole set of real numbers.

B) Suppose that $S = \mathbb{R}^N$.

If the payoff function is monotone on S, it does not have to take any supremum and when the optimal response sets are empty sets there cannot exist any Nash equilibrium. However, if the payoff function is not monotone on S, there exists at least one supremum which is also a maximum and hence the optimal response sets are nonempty. However, their values are dependent on strategies of other players and there does not have to exist any intersection point of these sets.

In the following example we introduce calculating Nash equilibria with concave payoff functions in a common economic model.

Example 6: (Cournot's model)

Suppose there are two companies producing lamb's wool. Denote y_1 an amount of wool produced by company 1 and y_2 an amount of wool produced by company 2. Both companies decide about these outputs simultaneously but separately, so they do not know what amount of wool the other one will choose. We suppose that the price of lamb's wool is determined by the market in dependence on produced amount but it is not implicitly influenceable by producers (companies). In this case we will suppose price to be an affine linear function of total output:

$$p = -a(y_1 + y_2) + b,$$

where $a, b > 0$. Let us mention that the price is decreasing when amount of outputs is increasing.

Both companies want to maximize their profits from produced wool. Clearly, earnings of company i are $p \cdot y_i$, $i = 1, 2$. Costs of company i c_i, $i = 1, 2$, are affine linear functions of amounts of outputs.

$$c_i = ky_i + l, \quad i = 1, 2,$$

where $k, l > 0$.

Hence payoff functions of companies are as follows

$$f_1(y_1, y_2) = (-a(y_1 + y_2) + b)\, y_1 - ky_1 - l,$$

$$f_2(y_1, y_2) = (-a(y_1 + y_2) + b)\, y_2 - ky_2 - l,$$

where $y_1, y_2 \geq 0$.

We will use general conditions for finding extremes:

$$\frac{\partial f_1(y_1, y_2)}{\partial y_1} = 0,$$

$$\frac{\partial f_2(y_1, y_2)}{\partial y_2} = 0.$$

After derivation we get

$$y_1 = \frac{-ay_2 + b - k}{2a},$$

$$y_2 = \frac{-ay_1 + b - k}{2a}.$$

Since we calculated the produced amounts maximizing profits as functions of produced outputs of the other company, we got the reaction curves (the optimal response sets) of both companies.

Hence we can calculate Nash equilibria as intersection points of these reaction curves.

$$-2ay_1 - a\frac{-ay_1 + b - k}{2a} + b - k = 0,$$

$$-a\frac{-ay_2 + b - k}{2a} - 2ay_2 + b - k = 0.$$

We get

$$y_1^* = \frac{b - k}{3a},$$

$$y_2^* = \frac{b - k}{3a}.$$

A vector (y_1^*, y_2^*) is a Nash equilibrium of both companies.

This game is in the economic theory called *Cournot's model*. Equilibrium in this model is called *Cournot equilibrium*. We can notice that players reach this equilibrium only if they expect exact amount of outputs produced by other players while maximizing their profits.

$\triangle$

Chapter 4

Conclusion

In this thesis On the Role of Linearity Assumption in Nash Games we discussed calculating of Nash equilibria in several types of games. The main analysis was structured in two parts. In the first part, Chapter 2, we introduced the game theory and further focused on a concept of Nash equilibria. In the second part, Chapter 3, we discussed 3 different types of games and analyze an existence and calculating Nash equilibria in them.

Application of the game theory is widely used in many different disciplines, for example philosophy, biology and many others. Its still increasing importance also appears in the economic theory where it is used for modeling common economic situations. Using the game theory economists can model situations on markets among economic actors. It helps describe such situations as monopoly, duopoly or oligopoly. One of the interests of the economic theory is to find equilibrium points in such economic situations. Since there is an assumption of rationality of economic actors, there is widely used also concept of Nash equilibria in non-cooperative situations. To maximize own profit is one of the convenient way how to get to Nash equilibrium if there exists some. However, we showed in this thesis that this behavior does not have to lead to the best outcome of the situation, in words of economists, to the common good. Recall the situation of Prisoner's dilemma. Companies in the market often get in similar situations where they have to choose between to trust to each other and not to betray to keep their credibility or to expect betrayal and to betray. In the reality it shows that in morally mature markets companies rather choose to trust and not to betray. This leads to question of sufficiency and convenience of rationality assumption for describing real situations on markets. However, it shows that the importance of other aspects, as morality or trust, instead of pure profit maximization is indispensable.

Appendix A

Basic Definitions and Theorems

Definition A.1. (*linear function*)
Let $f : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$, be a real-valued function. We say that f is *linear* if

$$f(\alpha x) = \alpha f(x)$$

$$f(x+y) = f(x) + f(y)$$

for every $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Definition A.2. Let $g : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$, be a real-valued function. We say that g is *affine linear* if there exist linear function $f : \mathbb{R}^N \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$g(x) = f(x) + c$$

for all $x \in \mathbb{R}^N$.

Theorem A.3. *Suppose $f : \mathbb{R}^N \to \mathbb{R}$ is an affine linear function. Then there exist coefficients $\alpha_1, ..., \alpha_N, \beta \in \mathbb{R}$ such that*

$$f(x_1, \dots, x_N) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N + \beta.$$

Definition A.4. (*piecewise affine linear function*)
Let $f : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$, be a real-valued function. We say that function f is *piecewise affine linear* if there exist subsets of $\mathbb{R}^N$, $M_1, ..., M_k$, $k \in \mathbb{N}$, mutually disjoint such that $\bigcup_{j=1}^k M_j = \mathbb{R}^N$ and f is affine linear on M_j, for every $j = 1, \dots, k$.

Definition A.5. (*convex and strictly convex function*)
Let $f : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$, be a real-valued function. We say that function f is *convex* on $\mathbb{R}^N$ if for every pair $(x_1, \dots x_N), (x'_1, \dots x'_N) \in \mathbb{R}^N$ and $\lambda \in [0,1]$ it holds

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x'_1, \dots, \lambda x_N + (1-\lambda)x'_N) \\ \le \lambda f(x_1, \dots, x_N) + (1-\lambda) f(x'_1, \dots x'_N). \end{aligned}$$

We say that f is *strictly convex* on $\mathbb{R}^N$ if for every pair $(x_1, \dots x_N), (x'_1, \dots x'_N) \in \mathbb{R}^N$, $(x_1, \dots x_N) \neq (x'_1, \dots x'_N)$, and $\lambda \in (0,1)$ it holds

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x'_1, \dots, \lambda x_N + (1-\lambda)x'_N) \\ < \lambda f(x_1, \dots, x_N) + (1-\lambda) f(x'_1, \dots x'_N). \end{aligned}$$

Definition A.6. (*concave and strictly concave function*)
We say that $f : \mathbb{R}^N \to \mathbb{R}$, $N \in \mathbb{N}$, is *(strictly) concave* on $\mathbb{R}^N$ if $-f$ is a (strictly) convex function.

Definition A.7. (*increasing function*)
We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *increasing* on $\mathbb{R}$ if for every pair $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, it holds that $f(x_1) < f(x_1)$.

Remark. Definitions of decreasing, non-increasing and non-decreasing function are analogous.

Definition A.8. (*constant function*)
We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *constant* on $\mathbb{R}$ if $f(x) = f(y)$ for all $x, y \in \mathbb{R}$.

Definition A.9. (*monotone and strictly monotone function*)
We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *monotone* (resp. *strictly monotone*) on $\mathbb{R}$ if it is non-increasing or non-decreasing (resp. increasing or decreasing) on $\mathbb{R}$.

Definition A.10. (*coercive function*)
Let S be an unbounded set and $f : S \to \mathbb{R}$. We say that f is *coercive* if it holds that

$$\mathop{lim}_{\substack{\|s\| \to \infty \\ s \in S}} f(s) = +\infty.$$

Definition A.11. (*convex set*)
The set $M \in \mathbb{R}^N$, $N \in \mathbb{N}$, is called *convex* if for all $x, y \in M$ and $\lambda \in [0, 1]$ it holds

$$\lambda x + (1 - \lambda) y \in M.$$

Theorem A.12 (Weierstrass)**.** *Let f be a continuous function on a compact interval $[a, b]$, $a, b \in \mathbb{R}$, $a \neq b$. Then there exist a maximum and a minimum of f on $[a, b]$.*

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